lecture

prove $(A \times B)^{-1} = B^{-1} \times A^{-1}$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 6 \\ 4 & 3 \end{bmatrix}$$

proof $A \times B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 7+8 & 6+6 \\ 21+16 & 19+12 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 37 & 30 \end{bmatrix}$$

$$(A \times B)^{-1} = \frac{1}{(15)(30)-(12)(37)} \begin{bmatrix} 30 & -12 \\ -37 & 15 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 30 & -12 \\ -37 & 15 \end{bmatrix}$$

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 6 \\ 4 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{21-24} \begin{bmatrix} 3 & -6 \\ -4 & 7 \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} 3 & -6 \\ -4 & 7 \end{bmatrix}$$

$$B^{-1} A^{-1} = \frac{1}{6} \begin{bmatrix} 3 & -6 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 30 & -12 \\ -37 & 15 \end{bmatrix}$$

2) Inverse $\Longrightarrow$ row elimination

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad A^{-1} = \frac{1}{3-2} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$[A / I] \xrightarrow{\text{عوامل}} [I / A^{-1}]$$

$$r_1 - r_2 \to r_2 \quad \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

$$2r_2 + r_1 \to r_1 \quad \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{array}\right]$$

$$-r_2 \to r_2 \quad \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & -1 & -1 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -1 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$= I \, A^{-1}$$

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Ex Find A^{-1}:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \quad \boxed{3\times 3}$$

$$|A| = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \end{array} = 1(5\times 8 - 0\times 3) - 2[2\times 8 - 1\times 3] + 3[2\times 0 - 1\times 5]$$

$$= 40 - 2(16-3) + 3(-5) = 40 - 26 - 15 = 1$$

$$A^{-1} = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \times \frac{1}{-1}$$

$$= - \begin{bmatrix} +(5\times8-0\times3) & -2(2\times8-0\times3) & +(2\times3-5\times3) \\ -(2\times8-1\times3) & +(1\times8-1\times3) & -(1\times3-2\times3) \\ +(2\times0-1\times5) & -(1\times2-2\times3) & +(1\times5-2\times2) \end{bmatrix}$$

$$= - \begin{bmatrix} 40 & -16 & 9 \\ -13 & 5 & 3 \\ -5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -9 & -1 \end{bmatrix}$$

Find A^{-1}:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = 2(1-0) - 0(-1+0) - 1(0+1)$$

$$= 2 - 1$$

$$= 1$$

$$A^{-1} = \begin{bmatrix} +(1-0) & -(0-0) & +(0+1) \\ -(-1-0) & +(2-1) & -(0-1) \\ +(0+1) & -(0-0) & +(2+0) \end{bmatrix}$$

$[A / I] \Longrightarrow [I / A^{-1}]$ Another

$r_1 + 2r_2 \to r_2$
$r_1 + 2r_3 \to r_3$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\cancel{\frac{r_1}{2} \to r_1}$ $r_3 + r_1 \to r_1$
$\cancel{\frac{r_2}{2} \to r_2}$ $r_3 + r_2 \to r_2$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{array}\right]$$

$\frac{r_1}{2} \to r_1$
$\frac{r_2}{2} \to r_2$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{array}\right]$$

A^{-1} =

Using Row Operations to find A^{-1}

$$A = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$-2r_1 + r_2 \to r_2$
$r_3 - r_1 \to r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$-r_2 + r_1 \rightarrow r_1 \quad 2r_2 + r_3 \rightarrow r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$9r_3 + r_1 \rightarrow r_1 \quad -3r_3 + r_2 \rightarrow r_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & 5 & 2 & 1 \end{array}\right]$$

$$-r_3 \rightarrow r_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & -13 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ -13 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$$

Ex Show that the Matrix A is Not Invertible by 2 different Method

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{vmatrix} = 1(20+2) - 6(10-1) + 4(4+4)$$

$$= 22 - 54 + 32 = 54 - 54 = 0$$

$\therefore \quad A^{-1} = \frac{1}{0} \Rightarrow \infty$

$-2r_1 + r_2 \rightarrow r_2$
$r_1 + r_3 \rightarrow r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

Not invertable

$r_2 + r_3 \rightarrow r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Not invertable !

Solve the system of linear equation by using the inverse of the glectriy

Ex Solve the system of equation by hand the inverse

$x + y = 3 \qquad 2x + y = 5$

Solution $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$\underset{2\times 2}{A} \quad \underset{2\times 1}{x} = \underset{2\times 1}{b}$$

$$A^{-1} A x = A^{-1} b \quad , \quad I x = A^{-1} b$$

$$x = A^{-1} b \Rightarrow A^{-1} = \frac{-1}{1} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} 3 & \cancel{+}5 \\ -6 & \cancel{-}5 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore x = 2 \qquad y = 1$

Solve the System of equation by using the inverse

$$x_1 + 2x_2 + 3x_3 = 5$$
$$2x_1 + 5x_2 + 3x_3 = 3$$
$$x_1 + \quad + 8x_3 = 17$$

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}}$$

$$Ax = b \qquad A^{-1} A x = A^{-1} b \quad \therefore x = A^{-1} b$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ \cancel{18}13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -40 & 16 & 9 \\ \cancel{-18}+13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} -200+201 \\ 65-66 \\ 25-23 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$ Powers of a Matrix

$$A^2 = A \times A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}$$

- $A^3 = A \times A \times A = A^2 \times A =$

$$= \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$A^{-3} = A^{-1} A^{-1} A^{-1} = (A^3)^{-1}$$

$$A^{-3} = \frac{1}{11 \times 41 - 30 \times 15} \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

laws of exponents :

if A is Square Matrix, r & s are integers

$$A^r A^s = A^{r+s} \qquad (A^r)^s = A^{rs}$$

Polynominal Expressions involving Matrices

if $p(x) = 2x^2 - 3x + 4$ & $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$

find $p(A) = ??$ then find A^{-1}

$$p(x) = 12x^2 - 3x + 4$$

$$p(A) = 12A^2 - 3A + 4I$$

$$p(A) = 2 \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3 \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1 & 5 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 8 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} -3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$p(A) = \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix} = 2A^2 - 3A + 4I$$

$$A^{-1}\left[B = 2A^2 - 3A + 4I \right]$$

$$A^{-1}B = 2A^{-1}A^2 - 3A^{-1}A + 4A^{-1}I$$

$$A^{-1}B = 2\underbrace{A^{-1}A}_{I}A - 3I + 4A^{-1}$$

$$A^{-1}B = 2A - 3I + 4A^{-1}$$

$$p(A) = \begin{bmatrix} 19 & 2 \\ 0 & 13 \end{bmatrix}$$

$$A^{-1}B - 4A^{-1} = 2A - 3I$$

$$A^{-1}\left[B - 4I \right] = 2A - 3I$$

$$A^{-1}\left\{\begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\} =$$

$$2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}^{-1} = \frac{1}{45}\begin{bmatrix} -5 & 4 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 & -2 \\ 0 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{45}\begin{bmatrix} -45 & 30 \\ 0 & 15 \end{bmatrix}$$

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What Conditions must b_1 & b_2 & b_3 Satisfy in order

For the system of equations :

$$x_1 + 2x_2 + 2x_3 = b$$

$$x_1 + \quad + x_3 = b_2$$

$$2x_1 + x_2 + 3x_3 = b_3$$

To Be Consist end ??

One Solution — more than 1 Solution

$r_1 - r_2 \rightarrow r_2$

$-2r_1 + r_3 \rightarrow r_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{array}\right]$$

$r_2 + r_3 \rightarrow r_3$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_1 - b_2 + b_3 - 2b_1 \end{array}\right]$$

$$x_2 + x_3 = b_1 - b_2 \qquad x_3 = t$$

$x_2 = -t + b_1 - b_2$ $\qquad$ $x_1 = -t + b_2$

Assignment $\qquad$ حل سؤال

$$x + 2y - 3z = 4$$
$$3x - y + 5z = 2$$
$$4x + y + (a^2 - 14)z = a + 2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2 - 14 & a+2 \end{array}\right] \Rightarrow \begin{array}{l} -3r_1 + r_2 \longrightarrow r_2 \\ -4r_1 + r_3 \longrightarrow r_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2 - 2 & a - 14 \end{array}\right]$$

No Sol	One Sol	More than 1 Sol
$a^2 - 16 = 0$	$R - \{4, -4\}$	$a^2 - 16 = 0$
$a - 4 \neq 0$		$a = 4$
$a \pm 4$		$a = 4$

$$-7y + 14z = -10 \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{8}{7} \\ \frac{10}{7} \\ 0 \end{bmatrix}$$

$7y = 10 - 14t$ $\qquad\qquad y = \frac{10}{7} - \alpha t$

$x + 2y - 3z = 4$

$x = \frac{-3v}{7} + 4t + 3t + 4$

$x = \frac{8}{7} + 7t$

12) $\underline{\underline{a}}$ $2x_1 - 3x_2 + 4x_3 - x_4 = 0$

$7x_1 + x_2 - 8x_3 + 9x_4 = 0$

$2x_1 + 8x_2 + x_3 - x_4 = 0$

$$\left[\begin{array}{cccc|c} 2 & -3 & 4 & -1 & 0 \\ 7 & 1 & -8 & 9 & 0 \\ 2 & 8 & 1 & -1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} 7r_1 - 2r_2 \rightarrow r_3 \\ r_1 - r_3 \rightarrow r_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & -3 & 4 & -1 & 0 \\ 0 & -23 & 44 & -25 & 0 \\ 0 & -11 & 3 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 2 & -3 & 4 & -1 & 0 \\ 0 & -22 & 44 & -25 & 0 \\ 0 & 0 & 45 & -275 & 0 \end{array}\right]$$